Fixed Point Theorems in Symmetric Metric Space for Faintly Compatible Maps

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Abstract
Using the CLRg property—which fully assumes the condition of closedness of the range of the involved map and has an advantage over the (E.A) property—this paper aims to prove some common coincidence and common fixed point theorems in the domain of symmetric metric space via faintly compatibility, under various contractive conditions. Consequently, authors are currently focusing on this property to generalise the results found in the literature. The findings here build upon and refine earlier, more well recognised findings in the field of fixed point theory.
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1. Introduction and Preliminaries:

Because the challenge of identifying fixed points of non-linear maps may be framed as a problem in many real-world applications of economics, engineering, physics, and applied science, fixed point theory is an important part of non-linear analysis. Commutativity, continuity, completeness, and an acceptable constraint on containment of ranges of concerned maps, as well as an appropriate contraction condition, are typical requirements of common fixed point theorems. As a result, studies in this area try to alleviate at least one of these diseases (for examples, see [6, 7], [9], [10], [11], [12], [14], [15], [17]). With this in mind, the idea of the Common Limit in the Range of g (CLRg) property in fuzzy metric space was first proposed by Sintunavarat et.al. [2]. Various fixed point theorems in cone metric space were shown by Manoj Kumar et.al [3] under various contractive situations by using this characteristic. Ciric [9] introduced the following contractive condition to establish fixed point theorem: $d(Tx,\ Ty) \leq \lambda\ M(x,\ y)$, where $0 \leq \lambda < 1$, for all $x,\ y \in$ X, where $M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), d(x,\ Ty),\ d(y,\ Tx)\}$, which is known as Ciric type contractive condition. Concepts of weak compatibility, compatibility, and non-compatibility stand on their own. As a matter of fact, maps that meet both contractive and noncontractive conditions might be considered to have faint compatibility, which does not imply compatibility in the presence of a shared fixed point (or coincidence point). For maps meeting Ciric type contractive condition and other kinds of contractive conditions via weakly compatible and common limit range properties, which fully purchase the criteria of continuity, this study aims to demonstrate coincidence and common fixed point theorems.

the extent to which the relevant map is enclosed, and it has a leg up on the (E.A.) property. This work expands upon and enhances the findings of previous works by addressing non-compatible discontinuous self mappings in non-complete symmetric metric space [4, 5] and [8].

The following basic definitions and results will be needed in the sequel.

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \longrightarrow E$ satisfies

(1) $0 \leq d(x, y)$, $\forall\ x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

Then d is called a *Symmetric metric* on X and (X, d) is called a *Symmetric metric space*.

Definition 1.2. Let f and g be two self maps defined on a set X. If $w = fu = gu$, for some u in X then u is called *coincidence point* of f and g, where w is called the *point of coincidence* of f and g.

Definition 1.3. [1] Let f and g be two self maps defined on a set X. Then f and g are said to be *weakly compatible* if they commute at their coincidence point.

Definition 1.4.[2] Let f and g be two self maps defined on a metric space (X, d). Then f and g are said to satisfy the *Common Limit in the Range of g(CLRg) property* if

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \qquad \text{for some } x \in X.$$

Definition 1.5. [13] A pair of self maps (f, g) of a metric space (X,d) is conditionally compatible if whenever the sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t$ and $\lim_{n\to\infty} (fgy_n, gfy_n) = 0$.

Definition 1.6. [6] A pair of self maps (f, g) of a metric space (X,d) is faintly compatible, iff f and g are conditionally compatible and f and g commute on a non empty subset of coincidence points whenever the set of coincidences is non empty.

2. Main Result:

The following two theorems extend the results of M.Aamri and D.El Moutawaki [4] and [8] into symmetric metric space.

Theorem2.1. *Letf andg be faintly compatible self mappingsofa symmetricmetric space* (X, d) *satisfying*

(1) *CLRg property.*

(2) $d(fx, fy) < \lambda max\{d(gx, gy),\ d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}, 0 \leq \lambda < 1.$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfies CLRg property,$\exists$ a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \qquad \text{forsome } x \in X.$$

Consider,

$d(fx_n, fx) < \lambda max\{d(gx_n, gx), d(fx_n, gx_n), d(fx, gx), d(fx_n, gx), d(fx, gx_n)\}$

Letting $n \to \infty$, we have $d(gx, fx) \leq \lambda max\{d(gx, gx), d(fx, gx), d(fx, gx)\})$

Hence $fx = gx$. Thus x is the coincidence point of f and g.

Let $z = fx = gx$. Since f and g are faintly compatible we obtain $fz = fgx = gfx = gz$

Now we prove that $fz = z$: Suppose that fz Gz.

Consider, $d(fz, z) = d(fz, fx) < d(fz, z)$, which is a contradiction.

Hence z=fz=gz. i.e. z is the common fixed point of f and g.

Uniqueness of the common fixed point can be proven easily.

Example: Let $X = [0, 10]$ and define $f, g : X \to X$ by

$$fx = \begin{cases} \frac{1+x}{2} & \text{if } x \leq 1 \\ \frac{x+7}{2} & \text{if } 1 < x \leq 10 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2 - x & \text{if } x \leq 1 \\ \frac{x+10}{2} & \text{if } 1 < x \leq 10 \end{cases}$$

Then f and g satisfy the contractivecondition (2) of Theorem 2.1. Also f and g satisfy CLRg property, let $\{x_n\}=\{1 - \frac{1}{n}\}$ be a sequence in X. *Then* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g1 = 1$ and the pair (f,g) are faintly compatible. Further f and g satisfies all the conditions of the above Theorem and haveauniquecommon fixedpointat $x = 1$.

The following Corollary is the consequence of above theorem:

Corollary2.2. *Letf andg betwofaintly compatibleself mappingsofasymmetric metric space* (X, d) *satisfying*

(1) CLRg property.

(2) $d(fx, fy) < \lambda\, max\{d(gx, gy), d(fy, gx), d(fx, gy)\}$ ∀ x G $y \in X$.

Then f and g have a unique common fixed point.

The next theorem involves the function $F : R^+ \to R^+$ which satisfies the following conditions:

F1: F is nondecreasing on R^+.

F2: $0 < F(t) < t$, for each $t \in (0, \infty)$.

Theorem 2.3. *Letf, g, S and T be self mappingsofasymmetricmetricspace* (X, d) *such that*

(1) $d(fx, gy) \leq F[max\{d(Sx, Ty), d(Sx, gy), d(Ty, gy)\}]$, $\forall\ x, y \in X^2$.

(2) (f, S) *satisfies CLRs property and* (g, T) *satisfies CLRt Property.*

(3) (f, S) *and* (g, T) *are faintly compatible.*

Then f,g,S and T have a unique common fixedpoint.

Proof. Since (f, S) satisfies CLRs property,∃ a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = Sx \qquad \text{for some } x \in X.$$

Similarly, (g, T) satisfies CLRt property implies ∃ a sequence $\{y_n\}$ suchthat

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = Ty \qquad \text{for some } y \in X.$$

Consider $d(fx_n, gy_n) \leq F\,[max\{d(Sx_n, Ty_n), d(Sx_n, gy_n), d(Ty_n, gy_n)\}]$

Now letting $n \to \infty$, we have $d(Sx, Ty) \leq F\,[d(Sx,Ty)]$. Therefore, Sx = Ty.
i.e. $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = Sx = Ty = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n$

Consider $d(fx, gy_n) \leq F\,[max\{d(Sx, Ty_n), d(Sx, gy_n), d(Ty_n, gy_n)\}]$

Now letting $n \to \infty$, we have $d(fx, Sx) \leq 0$. Hence $fx = Sx$.

Thus x is the coincidence point of f and S.

Since (f,S) are faintly compatible, we have $ffx = fSx = Sfx = SSx$.
Similarly, $Ty = gy$ implies y is the coincidence point of g and T. Again, since (g,T) are weakly compatible, we have $TTy = Tgy = gTy = ggy$.
Now we prove that $ffx = fx$:
Consider $d(ffx, fx) = d(ffx, gy) \leq F\,[max\{d(Sfx, Ty), d(Sfx, gy), d(Ty, gy)\}] \leq F\,[d(ffx, fx)]$
Then $ffx = fx = Sfx$. Hence fx is the common fixed point of f and S.
Similarly, gy is the common fixed point of g and T.
Since $fx = gy$, $z = fx$ is the common fixed point of f, g, S and T.

Uniqueness of the common fixed point can be proven easily.

Example: Let $X = [0, 10]$, define $f, g, S, T : X \to X$ by

$fx = 3$ if $x \leq 3$, $fx = 4$ if $x > 3$ and $Sx = 6 - x$ if $x \leq 3$, $Sx = 8$ if $x > 3$

$gx = 4$ if $x < 3$, $gx = 6-x$ if $x \geq 3$ and $Tx = (3+x)/2$ if $x \geq 3$, $Tx = 6$ if $x < 3$

Let $F : R^+ \to R^+$ be defined by $F(t) = t - 1$, then F is nondecreasing on R^+ and $0 < F(t) < t$, for each $t \in (0, \infty)$.

It can be verify that f , g, S, T satisfy the contractive condition (1) of Theorem 2.3. Let $\{x_n\}=\{3 - \frac{1}{n}\}$ *and* $\{y_n\}=\{3 + \frac{1}{n}\}$ *then* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = S3=3$ and $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = T3=3$.
Thus A *(f,S)* satisfies CLRs property and (g,T) satisfies CLRt property. Further f, g, S, T satisfies all the conditions of the above Theorem andhaveauniquecommon fixedpointat $x = 3$.

The above theorem can be reduced for three maps in the following way.

Corollary 2.4. *Let* f, g *and* S *be self mappings of a symmetricmetric space* (X, d) *such that*

(1) $d(fx, gy) \leq F\,[max\{d(Sx, Sy), d(Sx, gy), d(Sy, gy)\}]$, $\forall\ x, y \in X^2$.

(2) (f, S) *and* (g, S) *satisfies CLRs Property.*

(3) (f, S) *and* (g, S) *are faintly compatible.*

Then f,g and S have a unique common fixedpoint.

The following two theorems are the generalization of Theorems 2.2 of [5] and [8] respectively.

Theorem 2.5.*Let* (X, d) *be a symmetric metric space and let* f, g, S, T *be self maps on* (X, d) *such that*

(1) (f, S) *satisfies CLRs property and* (g, T) *satisfies CLRt Property.*

(2) $d(fx, gy) \leq hu_{x,y}(f, g, S, T)$ *where* $h \in (0, 1)$ *and*
$u_{x,y}(f, g, S, T) \in \{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\} \forall\, x, y \in X$

(3) (f, S) *and* (g, T) *are faintly compatible.*

Then f,g,S and T have a unique common fixedpoint.

Proof. Since (f, S) satisfies CLRs property, $\exists$ a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = Sx \quad \text{for some } x \in X.$$

Similarly, (g, T) satisfies CLRt property implies $\exists$ a sequence $\{y_n\}$ suchthat

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = Ty \quad \text{for some } y \in X.$$

Consider $d(fx_n, gy_n) \leq hu_{x_n,y_n}(f, g, S, T)$, where

$u_{x_n,y_n}(f, g, S, T) \in \{d(Sx_n, Ty_n), d(fx_n, Sx_n), d(gy_n, Ty_n), d(fx_n, Ty_n), d(gy_n, Sx_n)\}$

as $n \to \infty$, $u_{x_n,y_n} \in \{d(Sx, Ty)$ Hence, as $n \to \infty$ $d(Sx, Ty) \leq hd(Sx, Ty)$ which implies $Sx = Ty$.

Now we show that $fx = Sx$: Consider $d(fx, gy_n) \leq hu_{x,y_n}(f, g, S, T)$,

as $n \to \infty$, $u_{x,y_n} \in \{d(fx, Sx)\}$

we get $d(fx, Sx) \leq hd(fx, Sx)$ which implies $fx = Sx$

Similarly one can prove gy = Ty. Thus $fx = Sx = gy = Ty$.

Since (f,S) are faintly compatible, we have $ffx = fSx = Sfx = SSx$. Similarly (g,T) are faintly compatible implies $TTy = Tgy = gTy = ggy$.

Now we prove that $ffx = fx$: Consider $d(ffx, fx) = d(ffx, gy) \leq h\, d(ffx, fx)$ implies $ffx = fx = Sfx$. Therefore, fx is the common fixed point of f and S. Similarly, gy is the common fixed point of g and T.

Hence fx is the common fixed point of f, g, S and T

Uniqueness of the common fixed point can be proven easily.

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